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Edwards–Wilkinson surface over a spherical substrate: $1/f$ noise in the height fluctuations

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Abstract
We study the steady-state fluctuations of an Edwards–Wilkinson type surface with the substrate taken to be a sphere. We show that the height fluctuations on circles at a given latitude have the effective action of a perfect Gaussian $1/f$ noise, just as in the case of fixed radius circles on an infinite planar substrate. The effective surface tension, which is the overall coefficient of the action, does not depend on the latitude angle of the circles.

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1. Introduction

It was long ago that the voltage fluctuations of a current-carrying resistor were observed to have a power spectrum nearly proportional to the inverse of the frequency [1]. Since then it has been realized that this so-called $1/f$ noise can be detected in a large variety of phenomena. Examples range from the velocity fluctuations of moving interfaces [2], the number of stocks traded daily [3] to the spectra of speech and music [4]. Because of this apparent universality, the suggestion emerged that there might be a generic underlying mechanism. The problem has been extensively studied from a large number of viewpoints, but a general description has not emerged so far.

Antal *et al* have recently found [5] a new aspect of the $1/f$ noise. Namely, they proved that the mean square fluctuations (also called roughness) of periodic signals with Gaussian $1/f$ power spectra are distributed according to one of the extreme value probability distributions, the Fisher–Tippet–Gumbel (FTG) distribution [6, 7]. This means that if we can find physical systems where Gaussian $1/f$ noise is present and periodic boundary conditions are realized, then FTG statistics naturally emerges in a measurement, thus again suggesting a possible reason for the generality.

The periodic boundary conditions are, of course, strange restrictions when we look for a physical realization. One example was given in [5] for a system which meets the above requirements. If we consider a $d = 2$ Edwards–Wilkinson (EW) surface on an infinite planar substrate, and draw a circle on it with a given radius R, then the probability of a height configuration is given by the effective action of the desired Gaussian $1/f$ noise.

Here we consider the generalization of the above example to spherical substrates. A motivation for investigating this problem comes from the increasing amount of accessible data on the cosmic microwave background (CMB) radiation anisotropy. Recently CMB measurements of the Wilkinson Microwave Anisotropy Probe first year sky survey gave a detailed sky map on the temperature fluctuations of the radiation [8]. These fluctuations are believed to be Gaussian, and their angular power spectrum is an essential detail in the cosmological models. The statistics of the present data are still not perfect, and one could improve the reliability of conclusions by investigating the fluctuations on objects of various shapes. Here we investigate the case of circles on the sphere and we assume that the EW fluctuations provide a simple model of the temperature anisotropy. If we could prove that the fluctuations on circles of this spherical substrate have the same properties as in the planar case, then a comparison between the probability distribution of the measured data and the FTG distribution would tell us about the appropriateness of our assumption. We could also look at the decay of two point correlations of the data set, which is in fact a more direct way of checking the EW model assumption. However, we believe that the use of the distribution function might give an effective alternative method, since here we have not only an exponent, but also a universal scaling function to compare.

In our paper, we present the derivation of the probability density functional of a height configuration over the equator of the sphere. We conclude that the effective action is equal to that of the Gaussian $1/f$ noise, exactly as in the case of the planar substrate in [5]. Then we generalize our result to arbitrary circles with latitude angle ϑ_0.

2. The model

In the EW model, the probability $P[h] \sim \mathrm{e}^{-S[h]}$ of a height configuration $h(\vartheta, \varphi)$ defined on the 2-sphere S_2 can be given through the free-field effective action

$$S[h] = \frac{\sigma}{2} \int_{S_2} |\nabla h(\vartheta, \varphi)|^2 \, \mathrm{d}\Omega \tag{1}$$

where σ is the effective surface tension. In order to get the probability that h takes the value of some fixed h_0 on the equator, we have to integrate over field values at other points:

$$P[h_0] = Z^{-1} \int_{h(\frac{\pi}{2},\varphi)=h_0(\varphi)} \mathcal{D}h \, \mathrm{e}^{-S[h]} \tag{2}$$

where Z is a normalization coefficient, and equals the same integral without restriction on the equator. The functional integration is carried out on continuous functions.

3. Calculation of the effective action

We use the same trick as in [9] in order to get rid of the functional integration. Namely, we introduce the 'classical' solution h_c, and the new variable $\tilde{h} = h - h_c$. Since the action is quadratic in the field we can write it in the following form:

$$S[h] = S[h_c] + S[\tilde{h}]. \tag{3}$$

The 'classical' solution h_c is the function that solves the Laplace equation on the northern and southern hemispheres respectively, and on the equator satisfies the same boundary condition as the integration variable in (2):

$$\triangle h_c(\vartheta, \varphi) = 0 \qquad \text{if} \qquad \vartheta \neq \frac{\pi}{2}, \qquad h_c\left(\frac{\pi}{2}, \varphi\right) = h_0(\varphi). \tag{4}$$

Changing the integration variable h to $\tilde{h}$ we have

$$P\,[h_0] = Z^{-1}\,\mathrm{e}^{-S[h_c]} \int_{\tilde{h}(\frac{\pi}{2},\varphi)=0} \mathcal{D}\tilde{h}\,\mathrm{e}^{-S[\tilde{h}]}. \tag{5}$$

Since the new boundary condition is $\tilde{h}\left(\frac{\pi}{2},\varphi\right) = 0$, the integral is now independent of h_0 so it can be absorbed into Z.

In the 'classical' action we can perform an integration by parts:

$$S[h_c] = \frac{\sigma}{2}\int_0^{2\pi} \mathrm{d}\varphi \left\{ \lim_{x\to -0}\left[h_c \frac{\partial h_c}{\partial x}(1-x^2)\right] - \lim_{x\to +0}\left[h_c \frac{\partial h_c}{\partial x}(1-x^2)\right]\right\} \tag{6}$$

where we changed to variables $x = \cos\vartheta$ and φ for convenience.

In order to find $h_c(x,\varphi)$ we have to solve two Dirichlet problems on the two hemispheres with a common boundary condition at the equator. It is tempting to determine the continuous and so squared integrable solution to (4) as an expansion to spherical harmonics, which are eigenvectors of the Laplacian. This way the effect of the Laplacian on the expansion could easily be computed by changing the order of the sum and the Laplace operator. However, the latter step is not always correct, the Laplacian not being a bounded operator. We can convince ourselves that in the present case changing the order of the infinite sum and the Laplacian yields erroneous result. Instead, we use an expansion for which this step is allowed.

In spherical polar coordinates the Laplace equation reads

$$\frac{\partial}{\partial x}(1-x^2)\frac{\partial h_c}{\partial x} + \frac{1}{1-x^2}\frac{\partial^2 h_c}{\partial \varphi^2} = 0. \tag{7}$$

Putting the usual ansatz $h_c^n(x,\varphi) = Q^n(x)\,\mathrm{e}^{\mathrm{i}n\varphi}$ into (7), we obtain a second-order equation for Q^n. For $n \neq 0$, we give the general solution of this equation as the sum of the linearly independent solution of the following two equations of first order:

$$\frac{\mathrm{d}Q^n}{\mathrm{d}x} = \frac{\pm n}{1-x^2}Q^n. \tag{8}$$

The general solutions of (7) are

$$\begin{aligned} h_c^n(x,\varphi) &= \left[a_n^+\left(\frac{1-x}{1+x}\right)^{\frac{|n|}{2}} + a_n^-\left(\frac{1+x}{1-x}\right)^{\frac{|n|}{2}}\right]\mathrm{e}^{\mathrm{i}n\varphi} \qquad n \neq 0 \\ h_c^0(x,\varphi) &= a_0 + b\ln\frac{1-x}{1+x}. \end{aligned} \tag{9}$$

By prescribing regularity at the poles, only one term remains for each n:

$$\begin{aligned} h_c^n(x,\varphi) &= a_n z^{|n|}\,\mathrm{e}^{\mathrm{i}n\varphi} \qquad n \neq 0 \\ h_c^0(x,\varphi) &= a_0 \end{aligned} \tag{10}$$

where we introduced the notation

$$z = \sqrt{\frac{1-|x|}{1+|x|}} = \begin{cases} \tan\frac{\vartheta}{2} & \vartheta \leqslant \frac{\pi}{2} \\ \cot\frac{\vartheta}{2} & \vartheta \geqslant \frac{\pi}{2}. \end{cases} \tag{11}$$

We are ready to give the solution of the two Dirichlet problems (4):

$$h_c(z,\varphi) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} \hat{h}_0(n)\,\mathrm{e}^{\mathrm{i}n\varphi} z^{|n|} \tag{12}$$

where $\hat{h}_0(n)$ denote the Fourier coefficients of h_0. If $\sum_{n=-\infty}^{\infty} n^2|\hat{h}_0(n)|^2 < \infty$ then

$$\lim_{z\to 1}\frac{\partial h_c}{\partial z}(z,\varphi) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}|n|\hat{h}_0(n)\,\mathrm{e}^{\mathrm{i}n\varphi}. \tag{13}$$

Since

$$\frac{\partial h_c}{\partial x} = -\frac{\operatorname{sgn} x}{(1+|x|)\sqrt{1-x^2}}\frac{\partial h_c}{\partial z} \tag{14}$$

the effective action can be given in the following form:

$$S[h_c] = \sigma \int_0^{2\pi} \mathrm{d}\varphi\, h_0(\varphi) \lim_{z\to 1} z \frac{\partial h_c}{\partial z}(z,\varphi) = 2\sigma \sum_{n=1}^{\infty} n|\hat{h}_0(n)|^2 \tag{15}$$

where the last equality was obtained by using $\hat{h}_0(-n) = \hat{h}_0(n)^*$.

Thus we have arrived at our main result: the action describing the probability density of the height configurations on the equator is exactly the same as the one characterizing Gaussian $1/f$ noise.

3.1. Generalization to arbitrary circles

In the case the boundary values are prescribed not on the equator but for instance on the circle at latitude angle $0 < \vartheta_0 < \pi$, we introduce similar variables as before:

$$z = \sqrt{\frac{1-x}{1+x}} \qquad z_0 = \sqrt{\frac{1-x_0}{1+x_0}} \tag{16}$$

where $x_0 = \cos\vartheta_0$. The solutions of the two Dirichlet problems are

$$h_c^{\pm}(z,\varphi) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \hat{h}_0(n)\, \mathrm{e}^{\mathrm{i}n\varphi} \left(\frac{z}{z_0}\right)^{\pm|n|} \tag{17}$$

where h_c^+ (h_c^-) is defined for $\vartheta \leqslant \vartheta_0$ ($\vartheta \geqslant \vartheta_0$).

The integration in $S[h_c]$ is carried out along the circle at latitude angle ϑ_0, where the derivatives of the solutions are not continuous:

$$S[h_c] = \frac{\sigma}{2} \int_0^{2\pi} \mathrm{d}\varphi \left\{ \lim_{x\to -x_0} \left[h_c^- \frac{\partial h_c^-}{\partial x}(1-x^2) \right] - \lim_{x\to +x_0} \left[h_c^+ \frac{\partial h_c^+}{\partial x}(1-x^2) \right] \right\}. \tag{18}$$

Since $(1-x^2)\frac{\mathrm{d}z}{\mathrm{d}x} = -z$, the $-z_0^{-1}$ factor introduced by differentiating h_c^- with respect to z is cancelled after carrying out the limit. The same is valid for the h_c^+ term with the opposite sign. Therefore,we arrive at exactly the same effective action (15) as in the equatorial case.

4. Final remarks

The question naturally arises whether boundary conditions prescribed on objects different from circles yield the effective action of the Gaussian $1/f$ noise. A possible way of finding such objects is to apply conformal transformations to the solution obtained in the case of circles. These transformations leave the Laplace equation invariant but the boundary curves where the solutions on the two regions of the sphere are joined might take a more general shape. Because of the large variety of conformal transformations, finding the ones that result in the action of Gaussian $1/f$ noise requires more detailed analysis.

It would be worth performing the calculation on other surfaces, for instance, on a negative curvature surface besides the plane and the sphere. Furthermore, the generalization to surfaces of non-trivial topology would also be interesting.

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